

3-Generator Groups whose Elements Commute with Their Endomorphic Images Are Abelian

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ABSTRACT. A group in which every element commutes with its endomorphic images is called an E -group. Our main result is that all 3-generator E -groups are abelian. It follows that the minimal number of generators of a finitely generated non-abelian E -group is four.

1. Introduction and results

A group in which each element commutes with its endomorphic images is called an “ E -group”. It is known that an E -group is a 2-Engel group, and thus it is nilpotent, of nilpotent class at most 3. All abelian groups are trivially E -group, non-abelian E -groups of class 2 exist (see e.g., [3] and [4]) and examples of E -groups of class 3, asked by A. Caranti [7, Problem 11.46 a], are not known. The first examples of non-abelian E -groups are due to R. Faudree [5]. Faudree’s examples are 4-generator. Our main result is to prove that 4 is the minimal number of generator of a non-abelian E -group.

Theorem 1.1. *Every 3-generator E -group is abelian.*

The unexplained notation follows that of [1]. In [1, Theorem 1.1] we showed that a finite 3-generator E -group is nilpotent of class at most 2, and it is proved in [1, Theorem 1.3] that an infinite, 3-generator E -group is abelian. Thus, to prove Theorem 1.1, we are left with ruling out the case of a finite p -group, which is a 3-generator E -group of class 2. To prove the latter, the main ingredients are the following:

- (1) Theorems 2.2 and 2.5. In these theorems we classify 3-generator $p\mathcal{E}$ -groups by introducing the groups $G(p, r, t, [t_{ij}])$.
- (2) The result of Morigi [9] concerning p -groups with an abelian automorphism group, for p odd, (Theorem 4.1) and an adaptation to the case $p = 2$ (Proposition 4.2).
- (3) Lemma 2.7 in which we have proved a dichotomy for endomorphisms of a 3-generator pE -group: they are either central automorphism or their images are contained in the center.

2. Classification of 3-generator $p\mathcal{E}$ -groups

In [1, Theorem 2.10], a complete classification for all 3-generator $p\mathcal{E}$ -groups for $p > 2$ is given. Here we classify 3-generator $2\mathcal{E}$ -groups (Theorems 2.2 and 2.5, below). We also determine all $p\mathcal{E}$ -groups whose derived subgroups are cyclic (Theorem 2.4, below).

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Remark 2.1. We know that a finite pE -group is a $p\mathcal{E}$ -group [8]; but the converse is false in general ([1, Remark 2.2]). Besides pE -groups whose existence of class 3 is unknown, there exist $p\mathcal{E}$ -groups of class 3. Thanks to the `nq` package of W. Nickel which is available in `GAP` [6], one can construct the largest (with respect to the size) 2-Engel 9-generator group $G = \langle x_1, \dots, x_9 \rangle$ of exponent 27 with the following relations:

$$\begin{aligned} x_1^3 &= [x_2, x_3][x_4, x_5][x_6, x_7][x_8, x_9], & x_2^3 &= [x_1, x_3][x_4, x_6][x_5, x_8][x_7, x_9], & x_3^3 &= [x_1, x_2][x_4, x_7][x_5, x_9][x_6, x_8], \\ x_4^3 &= [x_1, x_5][x_2, x_6][x_3, x_9][x_7, x_8], & x_5^3 &= [x_1, x_4][x_2, x_8][x_3, x_7][x_6, x_9], & x_6^3 &= [x_1, x_7][x_2, x_9][x_3, x_5][x_4, x_8], \\ x_7^3 &= [x_1, x_8][x_4, x_9][x_3, x_6][x_2, x_5], & x_8^3 &= [x_1, x_9][x_3, x_4][x_2, x_7][x_5, x_6], & x_9^3 &= [x_1, x_6][x_3, x_8][x_2, x_4][x_5, x_7]. \end{aligned}$$

Now it can be easily seen by `GAP` [6], that we have $|G| = 3^{84}$, $|G'| = 3^{75}$, $|Z(G)| = 3^{39}$, $\exp(\frac{G}{G'}) = 3$, $G' = Z_2(G) \cong C_9^{36} \times C_3^3$ and $\Omega_1(G') = \gamma_3(G) = Z(G) \cong C_3^{39}$.

Since every commutator $[x_i, x_j]$ appears only once in the above relations, it follows that

$$\langle x_1^3, \dots, x_9^3 \rangle = \langle x_1^3 \rangle \times \dots \times \langle x_9^3 \rangle.$$

Therefore $|G^3| = |\langle x_1^3, x_2^3, \dots, x_9^3 \rangle G^3| = 3^{45}$ and so by regularity, $|\Omega_1(G)| = |G : G^3| = 3^{39}$. Hence $\Omega_1(G) = \gamma_3(G) = Z(G)$ and G is a $3\mathcal{E}$ -group of class 3. We were unable to show whether G is an E -group or not.

Theorem 2.2. *Let G be a non-abelian 3-generator $p\mathcal{E}$ -group, $\exp(\frac{G}{G'}) = p^r$, $\exp(G') = p^t$ and $(p > 2$ or $(p = 2$ and $\exp(G') \neq 2^r)$). Then $|G| = p^{3(r+t)}$ and G has the following presentation*

$$\begin{aligned} \langle x, y, z \mid x^{p^{r+t}} &= y^{p^{r+t}} = z^{p^{r+t}} = [x^{p^t}, y] = [x^{p^t}, z] = [y^{p^t}, x] = [y^{p^t}, z] = [z^{p^t}, x] = [z^{p^t}, y] = 1, \\ [x, y] &= x^{p^r t_{11}} y^{p^r t_{12}} z^{p^r t_{13}}, [x, z] = x^{p^r t_{21}} y^{p^r t_{22}} z^{p^r t_{23}}, [y, z] = x^{p^r t_{31}} y^{p^r t_{32}} z^{p^r t_{33}} \rangle, \end{aligned}$$

where $1 \leq t \leq r$ and $[t_{ij}] \in GL(3, \mathbb{Z}_{p^t})$. Moreover every group with the above presentation is a $p\mathcal{E}$ -group.

Proof. For the case $p > 2$, the proof is the same as the proof of Theorem 2.10 of [1]. For the other case, we need some modifications in the proof of the first case because of technical details. However we give the following proof covering both cases. By [1, Theorem 2.9], $\text{cl}(G) = 2$. Suppose that $\frac{G}{Z(G)} = \langle aZ(G) \rangle \times \langle bZ(G) \rangle \times \langle cZ(G) \rangle$, for some $a, b, c \in G$ such that $|aZ(G)| = |bZ(G)| = p^t$ and $|cZ(G)| = p^s$ for some integer s , $0 \leq s \leq t$. Then clearly $G' = \langle [a, b], [a, c], [b, c] \rangle$, $|[a, b]| \leq p^t$, $|[a, c]| \leq p^s$ and $|[b, c]| \leq p^s$. Therefore $|G'| \leq p^{t+2s}$. For all $x, y \in G$, we have $(xy)^{p^r} = x^{p^r} y^{p^r} [y, x]^{\frac{p^r(p^r-1)}{2}} = x^{p^r} y^{p^r}$. It follows that the map $x\Omega_r(G) \mapsto x^{p^r}$ is an isomorphism from $\frac{G}{\Omega_r(G)}$ to G^{p^r} . Thus $|G : \Omega_r(G)| = |G^{p^r}|$. Then $|G| = |\Omega_r(G)| |G^{p^r}| \leq |Z(G)| |G'|$ and so $|G : Z(G)| \leq |G'|$. Hence $p^{2t+s} \leq p^{t+2s}$ and $t \leq s$. It follows that $s = t$, $|G'| = |\frac{G}{Z(G)}| = p^{3t}$ and $G' = \langle [a, b] \rangle \times \langle [a, c] \rangle \times \langle [b, c] \rangle$. We have $G = \langle a, b, c \rangle$ (since $\frac{G}{G^{p^r} Z(G)} \cong C_p \times C_p \times C_p$).

Now, since $G^{p^r} \leq G'$ and $|G'| = |G : Z(G)| \leq |G : \Omega_r(G)| = |G^{p^r}|$, we have $G' = G^{p^r}$. By [1, Lemma 2.4], $\exp(G) = p^{r+t}$ and since $G' = G^{p^r}$ is an abelian group of order p^{3t} , it follows that $G^{p^r} = \langle a^{p^r}, b^{p^r}, c^{p^r} \rangle = \langle a^{p^r} \rangle \times \langle b^{p^r} \rangle \times \langle c^{p^r} \rangle$, and $|a| = |b| = |c| = p^{r+t}$. Also since $G^{p^r} = \langle a^{p^r} \rangle \times \langle b^{p^r} \rangle \times \langle c^{p^r} \rangle \leq \langle a^{p^t}, b^{p^t}, c^{p^t} \rangle$, it is not hard to see that $\langle a^{p^t}, b^{p^t}, c^{p^t} \rangle = \langle a^{p^r} \rangle \times \langle b^{p^r} \rangle \times \langle c^{p^r} \rangle$ and so

$$p^{3r} = |\langle a^{p^t}, b^{p^t}, c^{p^t} \rangle| \leq |G^{p^t}| \leq |\Omega_r(G)| \leq |Z(G)| = |G : G'| \leq p^{3r}.$$

It follows that $G^{p^t} = \Omega_r(G) = Z(G)$ and so $|G| = p^{3(r+t)}$. Since $G' = G^{p^r}$ there exists a 3×3 matrix $T = [t_{ij}] \in GL(3, \mathbb{Z}_{p^t})$ such that

$$[a, b] = a^{p^r t_{11}} b^{p^r t_{12}} c^{p^r t_{13}}, \quad [a, c] = a^{p^r t_{21}} b^{p^r t_{22}} c^{p^r t_{23}}, \quad [b, c] = a^{p^r t_{31}} b^{p^r t_{32}} c^{p^r t_{33}},$$

and every element of G can be written as $a^i b^j c^k$ for some $i, j, k \in \mathbb{Z}$, and

$$(a^i b^j c^k)(a^{i'} b^{j'} c^{k'}) = a^{i+i'-i'jp^rt_{11}-i'kp^rt_{21}-j'kp^rt_{31}} \\ b^{j+j'-i'jp^rt_{12}-i'kp^rt_{22}-j'kp^rt_{32}} c^{k+k'-i'jp^rt_{13}-i'kp^rt_{23}-j'kp^rt_{33}}$$

Now consider $\tilde{G} = \mathbb{Z}_{p^{r+t}} \times \mathbb{Z}_{p^{r+t}} \times \mathbb{Z}_{p^{r+t}}$ and define the following binary operation on \tilde{G} :

$$(i, j, k)(i', j', k') = (i + i' - i'jp^rt_{11} - i'kp^rt_{21} - j'kp^rt_{31}, \\ j + j' - i'jp^rt_{12} - i'kp^rt_{22} - j'kp^rt_{32}, k + k' - i'jp^rt_{13} - i'kp^rt_{23} - j'kp^rt_{33})$$

It is easy to see that, with this binary operation, \tilde{G} is a group and $G \cong \tilde{G}$. Now one can easily see that the group G has the required presentation. \square

Notation. For any prime number p , and integers r, t with $1 \leq t \leq r$ and $[t_{ij}] \in GL(3, \mathbb{Z}_{p^t})$, we write $G(p, r, t, [t_{ij}])$ to denote the group G with the presentation given in Theorem 2.2.

Lemma 2.3. *Let G be a finite nilpotent group of class 2. If G is 2-generator, then $|G| = |G'|^2 |Z(G)|$.*

Proof. Let $G = \langle a, b \rangle$, $H = \langle a \rangle Z(G)$ and $K = \langle b \rangle Z(G)$. Then H and K are normal subgroups of G . We see that $G = HK$ and $H \cap K = Z(G)$. If $|aZ(G)| = n$, then $[a, b]^n = 1$ and since $G' = \langle [a, b] \rangle$, $|G'|$ divides n . Therefore $|G'|$ divides $|\frac{H}{Z(G)}|$. Similarly $|G'|$ divides $|\frac{K}{Z(G)}|$. It follows that $|G'|^2 |Z(G)|$ divides $|G|$.

On the other hand, we have

$$|G : Z(G)| = |G : C_G(a) \cap C_G(b)| \leq |G : C_G(a)| |G : C_G(b)| \leq |G'|^2.$$

Hence $|G| = |G'|^2 |Z(G)|$. \square

Theorem 2.4. *Let G be a non-abelian $p\mathcal{E}$ -group with cyclic derived subgroup. Then G is isomorphic to $Q_8 \times C_2^n$, for some non-negative integer n .*

Proof. Since G is a p -group and G' is cyclic, there exist elements $a, b \in G$ such that $G' = \langle [a, b] \rangle$. Let $H = \langle a, b \rangle$, $\exp(\frac{G}{G'}) = p^r$ and $\exp(G') = p^t$. By Lemma 2.3,

$$|H'|^2 = |H : Z(H)| \leq |H : Z(G) \cap H| = |HZ(G) : Z(G)| \leq |G : Z(G)|.$$

Therefore $|G| \geq |G'|^2 |Z(G)|$. If $p > 2$ then by regularity, $|G| = |G^{p^r}| |\Omega_r(G)| \leq |G'| |Z(G)|$. This implies that G is abelian, a contradiction. Thus $p = 2$. Since G' is a cyclic 2-group and $a^{2^r}, b^{2^r} \in G'$ we have $\langle a^{2^r} \rangle \leq \langle b^{2^r} \rangle$ or $\langle b^{2^r} \rangle \leq \langle a^{2^r} \rangle$. We may assume that $a^{2^r} = b^{2^r s}$ for some integer s . It follows that $(ab^{-s})^{2^{r+1}} = 1$ and so $(ab^{-s})^2 \in Z(G)$. Thus $1 = [(ab^{-s})^2, b] = [a, b]^2$ and so $t = 1$. If $r \geq 2$ then $(ab^{-s})^{2^r} = 1$ and so $ab^{-s} \in Z(G)$ which implies that $[a, b] = 1$, a contradiction. Thus $r = 1$ and $G^2 = G'$. It follows that $a^2 = b^2 = [a, b]$ and so $H \cong Q_8$. Now we claim that $G = HC_G(H)$ and $C_G(H)$ is an elementary abelian 2-group. Assume on the contrary that there exists an element $g \in G$ such that $g \notin HC_G(H)$. Then $g^2 \neq 1$ and $g^2 = a^2 = b^2$ and so $(ga)^2 = [g, a]$. If $[g, a] = 1$, then $ga \in Z(G) \leq C_G(H)$ and $g \in HC_G(H)$, a contradiction. Therefore $[g, a] \neq 1$ and $[g, a] = [a, b]$. Similarly $[g, b] = [a, b]$. Then gab commute with a and b . Thus $g \in HC_G(H)$, a contradiction.

Next suppose that there exists $x \in C_G(H)$ such that $x^2 \neq 1$. We have $x^2 = a^2$ and $(xa)^2 = 1$. Then $xa \in Z(G)$ and so $1 = [xa, b] = [a, b]$ which is impossible. Hence our claim is proved. Also we have $H \cap C_G(H) = Z(H) = \langle a^2 \rangle$ and so $C_G(H) = \langle a^2 \rangle \times E$ for some elementary abelian 2-group E . Hence G is isomorphic to $H \times E$ and the proof is complete. \square

Now we complete the classification of 3-generator $p\mathcal{E}$ -groups.

Theorem 2.5. *Let G be a non-abelian 3-generator $2\mathcal{E}$ -group such that $\exp(\frac{G}{G'}) = \exp(G') = 2^r$. Then G is isomorphic to one of the following groups:*

- (i) $Q_8 \times C_2$

- (ii) $\langle x, y, z \mid x^4 = y^4 = [y, z] = 1, x^2 = z^2 = [x, y], (xz)^2 = y^2 \rangle$
- (iii) $\langle x, y, z \mid x^4 = z^4 = [y, z] = 1, x^2 = y^2 = [x, y], [x, z] = z^2 \rangle$
- (iv) $G(2, r, r, [t_{ij}])$ where $[t_{ij}] \in GL(3, \mathbb{Z}_{p^r})$.

Proof. Suppose that $\frac{G}{Z(G)} = \langle aZ(G) \rangle \times \langle bZ(G) \rangle \times \langle cZ(G) \rangle$, for some $a, b, c \in G$, where $|aZ(G)| = |bZ(G)| = 2^r$, $|cZ(G)| = 2^s$ and $0 \leq s \leq r$. If $s = 0$, then G' is cyclic and so by Theorem 2.4, G is isomorphic with $Q_8 \times C_2$. Therefore we may assume that $s \geq 1$. Clearly we have $G' = \langle [a, b], [a, c], [b, c] \rangle$. Since $a^{2^{r+s}}, b^{2^{r+s}} \in (G')^{2^s}$ and $(G')^{2^s}$ is a cyclic 2-group, we may assume that $a^{2^{r+s}} = b^{2^{r+s}k}$ for some integer k . It follows that $(ab^{-k})^{2^s} \in \Omega_r(G) \leq Z(G)$ and so $[a, b]^{2^s} = [a, ab^{-k}]^{2^s} = 1$. Therefore $\exp(G') \leq 2^s$. Thus $r = s$, $|\frac{G}{Z(G)}| = 2^{3r}$ and $|a| = |b| = |c| = 2^{2r}$. Since $2^{3r} = |G : Z(G)| \leq |G : \Omega_r(G)| \leq |G : G'| \leq 2^{3r}$ we have $G' = Z(G) = \Omega_r(G)$. Now the map $x\Omega_{r+1}(G) \mapsto x^{2^{r+1}}$ is an isomorphism from $\frac{G}{\Omega_{r+1}(G)}$ to $G^{2^{r+1}}$. It follows that

$$|G| = |\Omega_{r+1}(G)| |G^{2^{r+1}}| \leq |\Omega_{r+1}(G) : \Omega_r(G)| |\Omega_r(G)| |(G')^2| \leq 8|Z(G)| |(G')^2|$$

and so $|(G')^2| \geq 2^{3r-3}$. Suppose that $G' \cong C_{2^r} \times C_{2^u} \times C_{2^v}$ where $0 \leq v \leq u \leq r$. If $v = 0$ then $|(G')^2| = 2^{r+u-2} \leq 2^{2r-2}$. Therefore in this case $r = 1$, $|G| = 2^5$ and so by GAP [6] one can easily see that G has a presentation as in either (ii) or (iii). Then we may assume that $v \geq 1$ and $|(G')^2| = 2^{r+u+v-3}$ which implies that $u = v = r$, $|G'| = 2^{3r}$, $|G| = 2^{6r}$. It follows that $G' = \langle [a, b] \rangle \times \langle [a, c] \rangle \times \langle [b, c] \rangle$.

Now we claim that $G^{2^r} = G'$. If $r = 1$ then $|G| = 64$ and by GAP [6] it can be seen that there exist exactly four $2\mathcal{E}$ -groups G such that $\exp(\frac{G}{G'}) = \exp(G') = 2$ satisfying $G^2 = G'$. Thus we may assume that $r > 1$. We prove that $\langle a^{2^r}, b^{2^r}, c^{2^r} \rangle = \langle a^{2^r} \rangle \times \langle b^{2^r} \rangle \times \langle c^{2^r} \rangle$. If $a^{2^r m} b^{2^r n} c^{2^r l} = 1$ then $(a^{2m} b^{2n} c^{2l})^{2^r} = 1$ and so $a^{2m} b^{2n} c^{2l} \in Z(G)$. This implies that 2^r divide integers $2m, 2n$ and $2l$. Therefore all integers m, n and l are even and so $a^m b^n c^l \in \Omega_r(G) = Z(G)$. It follows that $a^{2^r m} = b^{2^r n} = c^{2^r l} = 1$. Hence $\langle a^{2^r}, b^{2^r}, c^{2^r} \rangle = \langle a^{2^r} \rangle \times \langle b^{2^r} \rangle \times \langle c^{2^r} \rangle \cong C_{2^r} \times C_{2^r} \times C_{2^r}$ and so $G^{2^r} = G'$. A proof similar to the last part of the proof of Theorem 2.2, gives that G is isomorphic to $G(2, r, r, [t_{ij}])$ for some matrix $[t_{ij}] \in GL(3, \mathbb{Z}_{p^r})$. This completes the proof. \square

Remark 2.6. It is not hard to see that groups (i), (ii) and (iii) in Theorem 2.5 are not E -groups.

Lemma 2.7. Let G be a finite 3-generator pE -group and $\alpha \in \text{End}(G)$.

- (i) If $\alpha \in \text{Aut}(G)$ then α is a central automorphism.
- (ii) If $\alpha \notin \text{Aut}(G)$ then $\text{Im} \alpha \leq Z(G)$, where $\text{Im} \alpha$ denotes the image of α .

Proof. Suppose that G is non-abelian, $\exp(G') = p^t$ and $\exp(\frac{G}{G'}) = p^r$. By Theorems 2.2 and 2.5 and Remark 2.6, there exist elements $a, b, c \in G$ such that $G = \langle a, b, c \rangle$, $|a| = |b| = |c| = p^{r+t}$, $G^{p^t} = Z(G) = \Omega_r(G)$, and

$$G^{p^r} = G' = \langle [a, b] \rangle \times \langle [a, c] \rangle \times \langle [b, c] \rangle, |[a, b]| = |[a, c]| = |[b, c]| = p^t.$$

Now we prove that $C_G(g) = \langle g \rangle Z(G)$ for each $g \in \{a, b, c\}$. By symmetry between a, b and c , it is enough to show this claim for $g = a$. Let $x \in C_G(a)$. Then there exist integers i, n, m and an element $w \in Z(G)$ such that $x = a^i b^n c^m w$. Since $[x, a] = 1$, we have $[b, a]^n [c, a]^m = 1$ and so $n \equiv m \equiv 0 \pmod{p^t}$. Therefore $x = a^i w_a$ for some $w_a \in Z(G)$, as required. Therefore $a^\alpha = a^i w_a$, $b^\alpha = b^j w_b$ and $c^\alpha = c^k w_c$, where $0 \leq i, j, k \leq p^t - 1$ and $w_a, w_b, w_c \in Z(G)$.

Now the proof may be completed by applying the same methods used in Section 4 of [3] concerning indecomposable pE -groups. But since these latter results are only stated for odd p in [3], we prefer to complete the proof for the reader's convenience.

From $[(ab)^\alpha, ab] = 1$ and $[(ac)^\alpha, ac] = 1$, it follows respectively that $i = j$ and $i = k$. Also from the equality $G^{p^r} = G'$, we have $a^{p^r} = [a, b]^s [b, c]^k [a, c]^l$ where s, k and l are integers. Thus $(a^\alpha)^{p^r} = [a^\alpha, b^\alpha]^s [b^\alpha, c^\alpha]^k [a^\alpha, c^\alpha]^l$ and we obtain $a^{p^r i} = a^{p^r i^2}$. Therefore $i^2 \equiv i \pmod{p^t}$ and so $i = 1$ or $i = 0$.

If $i = 1$, then α is a central automorphisms of G . If $i = 0$, then image α is in the center of G . This completes the proof. \square

3. A matrix formulation for a map to be an endomorphism of certain E -groups

Lemma 3.1 below, is somehow related to the results of [2], where dualities of the 3-dimensional vector space over the field with p -elements (only for odd prime p) are classified.

Notation. For a matrix $A = \begin{pmatrix} i_1 & j_1 & k_1 \\ i_2 & j_2 & k_2 \\ i_3 & j_3 & k_3 \end{pmatrix}$ we denote the matrix $\begin{pmatrix} k_3 & -k_2 & k_1 \\ -j_3 & j_2 & -j_1 \\ i_3 & -i_2 & i_1 \end{pmatrix}$ by \overline{A} . Also we denote by $\text{adj}(B)$ the adjoint of an square matrix B .

Lemma 3.1. *Let $G = G(p, r, t, [t_{ij}]) = \langle a, b, c \rangle$, where $p > 2$ or ($p = 2$ and $t \neq r$), $T = [t_{ij}] \in GL(3, \mathbb{Z}_{p^t})$ and let A be the above matrix. Then the map α defined by*

$$a^\alpha = a^{i_1} b^{j_1} c^{k_1} z_1, b^\alpha = a^{i_2} b^{j_2} c^{k_2} z_2, c^\alpha = a^{i_3} b^{j_3} c^{k_3} z_3,$$

where i_1, j_1, \dots, k_3 are integers and $z_1, z_2, z_3 \in Z(G)$, can be extended to an endomorphism of G if and only if the equality $TA = (\text{adj} \overline{A})T$ holds in the ring of matrices on \mathbb{Z}_{p^t} .

Proof. Since $\exp(G) = p^{r+t}$ and $\exp(G') = p^t$ we have $x^{p^{r+t}} = [x^{p^t}, y] = 1$ for all $x, y \in G$. Then α can be extended to an endomorphism of G if and only if

$$[a^\alpha, b^\alpha] = (a^\alpha)^{p^r t_{11}} (b^\alpha)^{p^r t_{12}} (c^\alpha)^{p^r t_{13}}, [a^\alpha, c^\alpha] = (a^\alpha)^{p^r t_{21}} (b^\alpha)^{p^r t_{22}} (c^\alpha)^{p^r t_{23}}, [b^\alpha, c^\alpha] = (a^\alpha)^{p^r t_{31}} (b^\alpha)^{p^r t_{32}} (c^\alpha)^{p^r t_{33}}.$$

Since $(xy)^{p^r} = x^{p^r} y^{p^r}$ for all $x, y \in G$ and $G^{p^r} = \langle a^{p^r} \rangle \times \langle b^{p^r} \rangle \times \langle c^{p^r} \rangle \cong C_{p^t} \times C_{p^t} \times C_{p^t}$, it follows that the following equality in the ring of matrices on \mathbb{Z}_{p^t} holds if and only if α can be extended to an endomorphism of G :

$$\begin{pmatrix} i_1 & i_2 & i_3 \\ j_1 & j_2 & j_3 \\ k_1 & k_2 & k_3 \end{pmatrix} \begin{pmatrix} t_{11} & t_{21} & t_{31} \\ t_{12} & t_{22} & t_{32} \\ t_{13} & t_{23} & t_{33} \end{pmatrix} = \begin{pmatrix} t_{11} & t_{21} & t_{31} \\ t_{12} & t_{22} & t_{32} \\ t_{13} & t_{23} & t_{33} \end{pmatrix} \begin{pmatrix} i_1 j_2 - j_1 i_2 & i_1 j_3 - j_1 i_3 & i_2 j_3 - j_2 i_3 \\ i_1 k_2 - k_1 i_2 & i_1 k_3 - k_1 i_3 & i_2 k_3 - k_2 i_3 \\ j_1 k_2 - k_1 j_2 & j_1 k_3 - k_1 j_3 & j_2 k_3 - k_2 j_3 \end{pmatrix}.$$

Hence by writing the above equality in the notation \overline{A} and adjoint the proof is complete. \square

4. Proof of the main result

Theorem 4.1. (The main result of [9]) *For p an odd prime, there exists no finite non-abelian 3-generator p -group having an abelian automorphism group.*

Although the above Theorem is false for $p = 2$ it is true for certain 2-groups.

Proposition 4.2. *There exists no finite non-abelian 3-generator 2-group G having an abelian automorphism group such that $\exp(G') = 2^t$, $\exp(G) = 2^{2t}$ and $t > 1$.*

Proof. The same proof as that of Theorem 4.1 works for this proposition. \square

Proof of Theorem 1.1. As we mentioned in Section 1, it is enough to show that every 3-generator pE -groups is abelian. Suppose, for a contradiction, that G is a non-abelian 3-generator pE -group. By Theorems 2.2 and 2.5 and Remark 2.6, there exists elements $a, b, c \in G$ such that $G = G(p, r, t, [t_{ij}]) = \langle a, b, c \rangle$, where $T = [t_{ij}] \in GL(3, \mathbb{Z}_{p^t})$.

Case I: $p > 2$; or $p = 2$ and $t \neq r$. Let $H = G(p, t, t, [t_{ij}]) = \langle x, y, z \rangle$.

We claim that every automorphism of H is central. If $\beta \in \text{Aut}(H)$, then

$$x^\beta = x^{i_1} y^{j_1} z^{k_1} z_1, y^\beta = x^{i_2} y^{j_2} z^{k_2} z_2, z^\beta = x^{i_3} y^{j_3} z^{k_3} z_3,$$

where $z_1, z_2, z_3 \in Z(H)$ and $i_1, j_1, \dots, k_3 \in \{0, \dots, p^t - 1\}$. If $A = \begin{pmatrix} i_1 & j_1 & k_1 \\ i_2 & j_2 & k_2 \\ i_3 & j_3 & k_3 \end{pmatrix}$ by Lemma 3.1 we

have $TA = (\text{adj} \bar{A})T$. Now we define the map α on G by

$$a^\alpha = a^{i_1} b^{j_1} c^{k_1}, b^\alpha = a^{i_2} b^{j_2} c^{k_2}, c^\alpha = a^{i_3} b^{j_3} c^{k_3}.$$

By Lemma 3.1, α can be extended to an endomorphism of G and by Lemma 2.7, α is a central automorphism or $\text{Im} \alpha \leq Z(G)$. If α is a central automorphism of G , then $a^{-1}a^\alpha \in Z(G)$ and so $a^{i_1-1}b^{j_1}c^{k_1}Z(G) = Z(G)$. Since $\frac{G}{Z(G)} = \langle aZ(G) \rangle \times \langle bZ(G) \rangle \times \langle cZ(G) \rangle$ and $|aZ(G)| = |bZ(G)| = |cZ(G)| = p^t$ we have $i_1 = 1, j_1 = 0, k_1 = 0$. Similarly $b^{-1}b^\alpha \in Z(G)$ and $c^{-1}c^\alpha \in Z(G)$. It follows that A is the identity matrix and so β is a central automorphism of H . If $\text{Im} \alpha \leq Z(G)$, then we similarly obtain that A is the zero matrix and so $\text{Im} \beta \leq Z(H)$, a contradiction.

Therefore all the automorphisms of H are central so that they fix the elements of $H' = Z(H)$. If $\varphi, \psi \in \text{Aut}(H)$, then $h^{\varphi\psi} = h^{\psi\varphi}$ for every $h \in \{x, y, z\}$. Hence $\text{Aut}(H)$ is abelian which contradicts Theorem 4.1 or Proposition 4.2 except when $p = 2$ and $t = 1$. In this case $|H| = 64$ and it can be easily checked by GAP [6] that there exist no $2\mathcal{E}$ -group of order 64 having an abelian automorphism group, a contradiction.

Case II: $p = 2$ and $t = r$. By Lemma 2.7 every automorphism of G is central and so $\text{Aut}(G)$ is abelian (since $G' = Z(G)$). As in **Case I** we reach to a contradiction. This completes the proof. \square

We end the paper with a result which generalizes [1, Theorem 2.9].

Theorem 4.3. *There exists no $p\mathcal{E}$ -group of class 3 such that $G = \langle x_1, x_2, \dots, x_n \rangle$ and for every $i \in \{1, 2, \dots, n\}$, the set $\{[x_i, x_j, x_k] \mid 1 \leq j < k \leq n, j \neq i \neq k\}$ is a linearly independent subset of the elementary abelian 3-group $\gamma_3(G)$.*

Proof. Suppose, for a contradiction, that G is a $p\mathcal{E}$ -group of class 3. Let $\exp(\frac{G}{G'}) = 3^r$ and $H = (G')^3 \gamma_3(G)$. Note that, by [1, Lemma 2.4], $[H, G] = H^{3^r} = 1$. Modulo H we have that

$$x_1^{3^r} = [x_1, x_2]^{m_2} [x_1, x_3]^{m_3} \cdots [x_1, x_n]^{m_n} \prod_{2 \leq i < j \leq n} [x_i, x_j]^{t_{ij}}$$

for some integers $m_2, m_3, \dots, m_n, t_{ij} \in \{-1, 0, 1\}$. Since $[x_1, x_1^{3^r}] = 1$, we have

$$\prod_{2 \leq i < j \leq n} [x_1, x_i, x_j]^{t_{ij}} = 1.$$

Now it follows from the hypothesis that $t_{ij} = 0$ for all i, j . Similarly, modulo H , we have $x_2^{3^r} = [x_2, x_1]^{m_1} [x_2, x_3]^{k_3} \cdots [x_2, x_n]^{k_n}$ where $m_1, k_3, \dots, k_n \in \{-1, 0, 1\}$. Since $[x_1^{3^r}, x_2] = [x_2^{3^r}, x_1]^{-1}$ we have $k_3 = m_3, \dots, k_n = m_n$. By a similar argument one can see that, modulo H ,

$$x_i^{3^r} = \prod_{j=1}^n [x_i, x_j]^{m_j} \text{ for all } i \in \{1, 2, \dots, n\}.$$

Therefore $[x_i, x_j]^{3^r} = \prod_{k=1}^n [x_i, x_k, x_j]^{m_k}$ for all $i, j \in \{1, 2, \dots, n\}$. It follows that

$$x_i^{3^{2r}} = (x_i^{3^r})^{3^r} = \prod_{j=1}^n [x_i, x_j]^{3^r m_j} = \prod_{j=1}^n \prod_{k=1}^n [x_i, x_k, x_j]^{m_j m_k} = 1$$

for all $i \in \{1, 2, \dots, n\}$. Hence $G^{3^{2r}} = 1$, contradicting [1, Lemma 2.4]. This completes the proof. \square

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